ETH zürich
Departement Informatik

Theoretische Informatik

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https://courses.ite.inf.ethz.ch/theo_inf_21

Exercises – Sheet 10

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Exercise 28

The *subset sum* problem (SUBSET-SUM for short) is the following decision problem: given a finite set $S = \{s_1, \ldots, s_m\}$ of natural numbers and a natural number t, decide if there exists a subset $U \subseteq S$ such that $\sum_{x \in U} x = t$. We seek to show that

$$3\text{SAT} \leq_p \text{SUBSET-SUM}$$

holds.

To get a reduction, we have to transform a formula in 3CNF to an instance of the subset sum problem. Let $\Phi = C_1 \wedge \ldots \wedge C_k$ be a formula in 3CNF over the set of variables $X = \{x_1, \ldots, x_n\}$. Without loss of generality, we suppose that every variable x_i occurs in at least one clause C_j and that no clause contains both a variable and its negation.

Now we construct two $(n+k)$-digit decimal numbers r_i and r'_i for every variable x_i and two $(n+k)$-digit decimal numbers s_j and s'_j for every clause C_j. The set S of the subset sum problem instance consists of exactly these $2(n+k)$ numbers.

The idea behind this construction is to assign each of the first n digits to a variable and each of the last k digits to a clause. We denote by $x[l]$ the l-th digit of $x \in S$, i. e., $x = x[1]x[2] \ldots x[n+k]$.

Then we can define the numbers in S, for all $1 \leq i \leq n$ and all $1 \leq j \leq k$, and the sum t as

$$r_i[l] = \begin{cases} 1 & \text{for } l = i \text{ or } (n+1 \leq l \leq n+k \text{ and } x_i \text{ occurs as a literal in } C_{l-n}) \\ 0 & \text{otherwise} \end{cases}$$

$$r'_i[l] = \begin{cases} 1 & \text{for } l = i \text{ or } (n+1 \leq l \leq n+k \text{ and } \overline{x_i} \text{ occurs as a literal in } C_{l-n}) \\ 0 & \text{otherwise} \end{cases}$$

$$s_j[l] = \begin{cases} 1 & \text{for } l = n+j \\ 0 & \text{otherwise} \end{cases}$$

$$s'_j[l] = \begin{cases} 2 & \text{for } l = n+j \\ 0 & \text{otherwise} \end{cases}$$

$$t[l] = \begin{cases} 1 & \text{for } 1 \leq l \leq n \\ 4 & \text{otherwise} \end{cases}$$

Prove that this reduction satisfies the following: Φ is satisfiable if and only if there exists a subset $U \subseteq S$ such that $\sum_{x \in U} x = t$. **10 points**

Exercise 29

Let Σ be an arbitrary alphabet such that $|\Sigma| \geq 2$. Let $w = a_1 a_2 \dots a_m$ be a word such that $a_i \in \Sigma$, for $1 \leq i \leq m$. We say that a word $x = x_1 x_2 \dots x_n$ is a *subsequence* of w if $n \leq m$ and there exist indices $i_1 < i_2 < \dots < i_n$ such that $x_j = w_{i_j}$ holds for all $1 \leq j \leq n$. Intuitively, all symbols of x appear in w in the same order, but not necessarily consecutively.

A word w over Σ is *Davenport-Schinzel sequence* of order d if it satisfies the following two conditions:

(i) no two consecutive symbols in w are equal,

(ii) for all $a, b \in \Sigma$ with $a \neq b$, there exists no subsequence $(ab)^{1+(d/2)}$ (for even d) or $(ab)^{(d+1)/2}a$ (for odd d) in w.

Provide a propositional formula in CNF that describes if there exists a Davenport-Schinzel sequence of order 2 and of length m over a given alphabet Σ with $|\Sigma| = k \geq 2$. **10 points**

Submission: On Friday, December 10, 2021, by 11:15 at the latest, as a legible PDF via e-mail directly to the respective teaching assistant.