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Theoretische Informatik

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Exemplary Solutions – Sheet 1

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Solution to Exercise 1

(a) If a word of length $n \in \mathbb{N}$ does not contain the subword 01, then it has the form $1^l 0^m$ for some $l, m \in \mathbb{N}$ such that $l + m = n$. Given a fixed n, it follows that $0 \leq l \leq n$ and m is uniquely determined by the choice of l. Hence, there are exactly $n+1$ such words of length n.

(b) Let $\Sigma = \{0, 1\}$ and $n \in \mathbb{N}$. We denote by L_n the set of words in Σ^n that do not contain the subword 00 and define $N(n) = |L_n|$. We seek to specify $N(n)$ recursively. It clearly holds that $N(0) = 1$ because there is a single word λ of length 0 over $\{0, 1\}$ and this word does not contain the subword 00. Moreover, $N(1) = 2$ because none of the words 0 and 1 contains the subword 00.

Let us now consider a word $w \in L_{n+1}$ such that $n \geq 1$. Then we can express w as $w = xab$ where $a, b \in \{0, 1\}$ and $x \in L_{n-1}$. If $b = 1$, then xa is an arbitrary word in L_n. If $b = 0$, then $a = 1$ must hold and x is an arbitrary word in L_{n-1}. Hence, we derive the following recurrence for $N(n)$:

$$N(n+1) = N(n) + N(n-1)\,.$$

This is the well-known recurrence for Fibonacci numbers, which are defined as $F(0) = 0$, $F(1) = 1$, and $F(n) = F(n-1) + F(n-2)$, for all $n \geq 2$. Due to the initial conditions $N(0) = 1$ and $N(1) = 2$, we conclude that

$$N(n) = F(n+2)\,.$$

The sequence of Fibonacci numbers can be expressed by the closed formula of Moivre-Binet:

$$F(n) = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}}\,,$$

where $\varphi = \frac{1+\sqrt{5}}{2}$. Hence, we obtain the following closed expression for $N(n)$:

$$N(n) = \frac{\varphi^{n+2} - (1-\varphi)^{n+2}}{\sqrt{5}}\,.$$

(c) In subtask (a), we have observed that every word of length n that does not contain the subword 01 has the form $1^l 0^m$ for some $l + m = n$. First, let $n \geq 2$. The only words of the form $1^l 0^m$ that do not contain the subword 00 are 1^n and $1^{n-1}0$. Hence, there are exactly 2 such words for every $n \geq 2$. For $n \leq 1$, all words satisfy the condition. Hence, there are one word of length 0 and two words of length 1 that satisfy the condition.

Solution to Exercise 2

(a) Let Σ be an alphabet and let $u, v \in \Sigma^*$. Then there exist $k, l \in \mathbb{N}$ such that $u = u_1 u_2 \dots u_k$ and $v = v_1 v_2 \dots v_l$, where $u_1, \dots, u_k, v_1, \dots, v_l \in \Sigma$. Hence, we have

$$uv = u_1 u_2 \dots u_k v_1 v_2 \dots v_l$$

and thus

$$\begin{aligned}(uv)^{\mathsf{R}} &= v_l \dots v_2 v_1 u_k \dots u_2 u_1 \\ &= v^{\mathsf{R}} u^{\mathsf{R}},\end{aligned}$$

because $v^{\mathsf{R}} = v_l \dots v_2 v_1$ and $u^{\mathsf{R}} = u_k \dots u_2 u_1$.

(b) This statement is false and we provide a counterexample. Let $L_1 = \{\lambda\}$ and $L_2 = \{a\}^*$. Then $L_2 - L_1 = \{a\}^+$, and thus $L_2(L_2 - L_1) = \{a\}^*\{a\}^+ = \{a\}^+$. However, $(L_2)^2 = \{a\}^*\{a\}^* = \{a\}^*$ and $L_2 L_1 = \{a\}^*\{\lambda\} = \{a\}^*$, and thus $(L_2)^2 - (L_2 L_1) = \{a\}^* - \{a\}^* = \emptyset$.

Solution to Exercise 3

(a) We choose $\Sigma = \{a\}$,

$$L_1 = \{a, aa, \dots, a^k\} = \{a^i \mid 1 \leq i \leq k\}$$

and

$$L_2 = \{\lambda, a\}.$$

Then

$$\begin{aligned}L_1 \cdot L_2 &= L_1 \cdot \{\lambda\} \cup L_1 \cdot \{a\} \\ &= L_1 \cup \{a^i a \mid 1 \leq i \leq k\} \\ &= L_1 \cup \{a^i \mid 2 \leq i \leq k+1\} \\ &= \{a^i \mid 1 \leq i \leq k+1\},\end{aligned}$$

and thus $|L_1 L_2| = k + 1$.

(b) We choose $\Sigma = \{a, b\}$,

$$L_1 = \{a^i \mid 1 \leq i \leq k\}$$

and

$$L_2 = \{b^i \mid 1 \leq i \leq l\}\,.$$

Then

$$L_1L_2 = \{a^ib^j \mid 1 \leq i \leq k \text{ and } 1 \leq j \leq l\}\,,$$

and thus $|L_1L_2| = k \cdot l$.

(c) To achieve the same cardinality of $L_1 \cdot L_2$ over a singleton alphabet, we must make sure that each pair of words in L_1 and L_2 yields a unique word. We can achieve that by choosing the lengths of the words in L_1 so that no word $v \in L_1$ can become the next longer word in L_1 after appending an arbitrary word $w \in L_2$. The idea is to keep the lengths of the words in L_2 as small as possible and have the lengths of the words in L_1 sufficiently far apart. Let $\Sigma = \{a\}$. We choose

$$L_1 = \{a^{li} \mid 0 \leq i \leq k-1\}$$

and

$$L_2 = \{a^j \mid 0 \leq j \leq l-1\}\,.$$

Then

$$L_1L_2 = \{a^{li}a^j \mid 0 \leq i \leq k-1 \text{ and } 0 \leq j \leq l-1\}\,.$$

All words in L_1L_2 have pairwise distinct lengths because the length of all words in L_1 is a multiple of l and all words in L_2 are shorter than l. Hence, $|L_1L_2| = k \cdot l$.