ETH zürich
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Theoretische Informatik

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https://courses.ite.inf.ethz.ch/theo_inf_21

Exemplary Solutions – Sheet 3

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Solution to Exercise 7

By Definition 2.64 in the textbook, a word $w \in \{0,1\}^*$ is said to be random if $K(w) \geq |w|$. In the following, we use a simple counting argument to show that at least half of the words $w \in \{0,1\}^*$ with $0 \leq |w| \leq n$ actually satisfy $K(w) \geq n$. This immediately implies the statement to be proved.

There are $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ words over $\{0,1\}$ of length at most n.

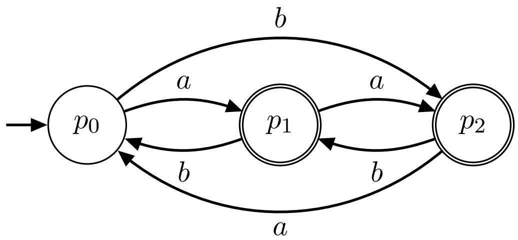
On the other hand, there are $\sum_{i=0}^{n-1} 2^i = 2^n - 1$ words over $\{0,1\}$ of length at most $n-1$. Hence, there are at most $2^n - 1$ binary representations of programs with length at most $n-1$. Because two distinct words can only be produced by two distinct programs, there are at most $2^n - 1 \leq \frac{1}{2} \cdot (2^{n+1} - 1)$ words with Kolmogorov complexity at most $n-1$. This immediately implies our claim.

Solution to Exercise 8

(a) We have

$$\begin{aligned} L_1 &= \{w \in \{a,b\}^* \mid (|w|_a + 2 \cdot |w|_b + 1) \bmod 3 \neq 1\} \\ &= \{w \in \{a,b\}^* \mid (|w|_a - |w|_b) \bmod 3 \neq 0\}. \end{aligned}$$

The following finite automaton accepts the language L_1:



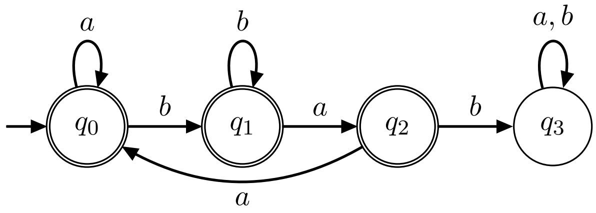
This automaton simply counts the input symbols modulo 3 where, according to the required condition, every a contributes positively and every b contributes negatively. This yields the classes

$$\mathrm{Kl}[p_i] = \{w \in \{a,b\}^* \mid (|w|_a - |w|_b) \bmod 3 = i\}$$

for $i \in \{0,1,2\}$.

(b) The following finite automaton accepts the language

$$L_2 = \{w \in \{a,b\}^* \mid w \text{ does not contain the subword } bab\}\,.$$



The automaton contains the same transitions as the automaton searching for the pattern bab in the input. When in state q_i, a prefix of length i of the pattern has already been found at the current position in the input. Because the automaton is supposed to accept all words that do not contain the pattern, all states except for q_3 are accepting.

There are the following classes for the states:

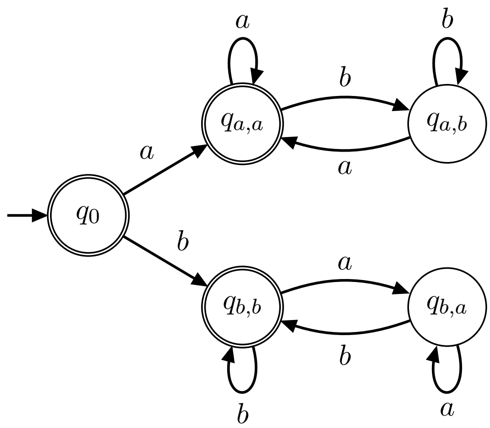
$$\begin{aligned}
\mathrm{Kl}[q_3] &= \{a,b\}^* - L_2\,,\\
\mathrm{Kl}[q_2] &= \{xba \mid x \in \{a,b\}^*\} \cap L_2\,,\\
\mathrm{Kl}[q_1] &= \{xb \mid x \in \{a,b\}^*\} \cap L_2\,,\\
\mathrm{Kl}[q_0] &= \{a,b\}^* - \bigcup_{i=1}^{3} \mathrm{Kl}[q_i]\,.
\end{aligned}$$

We remark that it is often useful to choose a suitable order of describing the classes.

Solution to Exercise 9

(a) The following finite automaton accepts the language

$$L = \{w \in \{a,b\}^* \mid w \text{ contains the subword } ab \text{ as many times as the subword } ba\}\,.$$



When reading the symbols in w from left to right, the subword ab corresponds to switching from a to b and, analogously, the subword ba corresponds to switching from b to a. Hence, w contains the subword ab as many times as the subword ba if and only if w is the empty word λ or w starts and ends with the same symbol. The latter condition is also satisfied for $w = a$ and for $w = b$. If w starts with b, then the automaton checks in the uppermost two states whether the last symbol read so far matches the first symbol. If w starts with a, then the automaton proceeds analogously in the bottommost two states.

(b) There are the following classes for the states:

$$\begin{aligned} \mathrm{Kl}[q_0] &= \{\lambda\} \text{ and} \\ \mathrm{Kl}[q_{x,y}] &= \{w \in \{a,b\}^* \mid w \text{ starts with } x \text{ and ends with } y\} \text{ for all } x, y \in \{a,b\}. \end{aligned}$$