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Theoretische Informatik

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Exemplary Solutions – Sheet 10

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Solution to Exercise 28

We consider the reduction from the task statement and show that the formula Φ is satisfiable if and only if there exists a subset $U \subseteq S$ such that $\sum_{x \in U} x = t$.

Let α be a satisfying assignment for Φ. We use it to construct a subset U_α as follows: For every variable x_i with $\alpha(x_i) = 1$, we choose $r_i \in U_\alpha$, for every variable x_i with $\alpha(x_i) = 0$, we choose $r'_i \in U_\alpha$. For every clause C_j in which α satisfies all three literals, we also include s_j in U_α. Analogously, we include s'_j in U_α if α satisfies exactly two literals in C_j, and we include both s_j and s'_j in U_α if α satisfies exactly one literal in C_j. It remains to show that $\sum_{x \in U_\alpha} x = t$ holds for the set U_α that we just defined.

We first observe that no carry can ever emerge during addition. For all $1 \leq i \leq n$, the i-th digit is only equal to 1 in the two numbers r_i and r'_i while it is equal to 0 in all other numbers. For all $1 \leq j \leq k$, the $(n + j)$-th digit is only nonzero in at most five numbers, namely it is equal to 1 in at most three numbers r_i or r'_i, where x_i occurs positively or negatively in C_j, and it is equal to 1 and 2 in the two numbers s_j and s'_j. Consequently, the $(n + j)$-th digit of the sum can never exceed 6. Hence, the individual digits can be considered separately in the proof.

We first analyze the i-th digit, for $1 \leq i \leq n$. Because α assigns x_i either with $\alpha(x_i) = 0$ or $\alpha(x_i) = 1$, exactly one of the two numbers r_i and r'_i is included in U_α. Because the i-th digit is equal to 0 in all other numbers, the i-th digit of the sum is equal to $t[i] = 1$. If α satisfies exactly l literals in the clause C_j, for some $l \in \{1, 2, 3\}$, then the corresponding numbers r_i and r'_i for these literals contribute l to the $(n + j)$-th digit of the sum. Adding the numbers s_j and s'_j according to the above rule yields the sum equal to $t[n + j] = 4$. Hence, U_α satisfies the condition that $\sum_{x \in U_\alpha} x = t$.

In the other direction, let us assume that U is a subset of S such that $\sum_{x \in U_\alpha} x = t$. We use it to construct an assignment α_U for Φ as follows: If $r_i \in U$, then we set $\alpha_U(x_i) = 1$, if $r'_i \in U$, then we set $\alpha_U(x_i) = 0$. For every $i \in \{1, \ldots, n\}$, exactly one of these cases must occur because of $t[i] = 1$. Hence, α_U is well-defined. Now we show that α_U satisfies the formula Φ. Because $t[n + j] = 4$, but the $(n + j)$-th digits of the numbers s_j and s'_j add up to at most 3, for every $1 \leq j \leq k$, one of the numbers r_i or r'_i with $r_i[n + j] = 1$ or $r'_i[n + j] = 1$ must be included in U. If this number is r_i, then $\alpha_U(x_i) = 1$ was set. By construction, the variable x_i occurs positively in C_j (because $r_i[n + j] = 1$) and thus the clause C_j is satisfied by α_U. If the number r'_i with $r'_i[n + j] = 1$ is included in U, then $\alpha_U(x_i) = 0$ was set. By construction, the variable x_i occurs negatively in C_j (because

$r'_i[n+j] = 1$) and thus the clause C_j is again satisfied by α_U. Hence, α_U is a satisfying assignment for Φ.

Solution to Exercise 29

Let Σ be an alphabet with $|\Sigma| = k \geq 2$ and let $w = a_1 a_2 \ldots a_m$ be a word of length m over Σ. To derive a formula in CNF that is satisfiable if and only if w is a Davenport-Schinzel sequence of order 2 over Σ, we use the variables $X_{a,j}$ for all $a \in \Sigma$ and all $1 \leq j \leq m$. Assigning 1 to the variable $X_{a,j}$ means that the symbol a occurs at the j-th position in w, i.e., $w_j = a$. Analogously, assigning 0 to the variable $X_{a,j}$ means that $w_j \neq a$.
Then the condition (i) can be expressed by the subformula:

$$F_{(\mathrm{i})} = \bigwedge_{a \in \Sigma} \bigwedge_{1 \leq j \leq m-1} \left(\overline{X_{a,j}} \vee \overline{X_{a,j+1}}\right).$$

For all possible symbols and all pairs of consecutive positions in w, $F_{(\mathrm{i})}$ prevents the same symbol from occurring at these positions.
To express the condition (ii) by a subformula, we consider four positions $1 \leq j_1 < j_2 < j_3 < j_4 \leq m$ in w and two arbitrary distinct symbols a and b from Σ and rule out the pattern $abab$ at these positions. For fixed positions, the following must hold:

$$\neg\left(X_{a,j_1} \wedge X_{b,j_2} \wedge X_{a,j_3} \wedge X_{b,j_4}\right).$$

This is equivalent to the clause

$$Z_{a,b,j_1,j_2,j_3,j_4} = \left(\overline{X_{a,j_1}} \vee \overline{X_{b,j_2}} \vee \overline{X_{a,j_3}} \vee \overline{X_{b,j_4}}\right).$$

This yields the following subformula for the condition (ii):

$$F_{(\mathrm{ii})} = \bigwedge_{a,b \in \Sigma, a \neq b} \; \bigwedge_{1 \leq j_1 < j_2 < j_3 < j_4 \leq m} Z_{a,b,j_1,j_2,j_3,j_4}$$

and we derive the following formula in CNF to express the Davenport-Schinzel condition of order 2 for the word w:

$$F_{(\mathrm{i})} \wedge F_{(\mathrm{ii})}.$$

Because we are not supposed to provide a formula for a fixed word w, but rather one expressing if a Davenport-Schinzel sequence over Σ of a given length m exists, we have to make sure that the assignment of the variables corresponds to a word.
The subformula

$$F_{\mathrm{Wort},1} = \bigwedge_{1 \leq i \leq m} \; \bigwedge_{a,b \in \Sigma, a \neq b} \left(\overline{X_{a,i}} \vee \overline{X_{b,i}}\right)$$

states that no two distinct symbols occur at the same position in the word. Furthermore, the subformula

$$F_{\mathrm{Wort},2} = \bigwedge_{1 \leq i \leq m} \; \bigvee_{a \in \Sigma} X_{a,i}$$

states that a symbol occurs at every position in the word.
Finally, we obtain the formula

$$F_{(\mathrm{i})} \wedge F_{(\mathrm{ii})} \wedge F_{\mathrm{Wort},1} \wedge F_{\mathrm{Wort},2}.$$